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POSSIBLE FOUNDATIONS OF CERTAIN SIMPLE MECHANICAL ANALOGIES IN HYDROMECHANICS

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Consideration is given to an approach qualitatively illustrating a formal substantiation of the analogies between Eulerian hydrodynamic systems and finite-dimensional mechanical systems of the oscillator type.

The equation (1) considered in [1]

$$\ddot{\varphi} + 2\beta\,(1 + \alpha\dot{\varphi}^2)\,\dot{\varphi} + \omega_0\,[1 + \xi_1\,(t) + a\cos\omega t]\sin\varphi = \xi_2\,(t)$$

of the oscillations $\varphi(t)$ of a pendulum with random perturbations of the suspension axis $\xi_1(t)$ and $\xi_2(t)$ is qualified by the authors as an "analogy" not related to or at least not following from the hydrodynamic equations that describe turbulent jet flow. However, without focusing attention on a specific form of the equation we can demonstrate a possible source of origin of its structure. Indeed, it is well known that the Eulerian infinite-dimensional representation of hydrodynamic equations in certain procedures can be reduced to a set of finite-dimensional representations. Such procedures, for example, are passage to a Lagrangian description, Galerkin-type spectral expansions, discrete-vortex modeling, etc. The characteristics of flow in such approaches are described, as a rule, by the systems of ordinary differential equations for finite-dimensional modes of these representations.

To make the representation more lucid let us consider, for example, two-dimensional flow with an Eulerian velocity field $u(x, y, t)$, $v(x, y, t)$ described by the Euler or Navier–Stokes equations. The Lagrangian description of such a flow has the form

$$\dot{X} = u\,(X, Y, t)\,, \quad \dot{Y} = v\,(X, Y, t)\,, \tag{1}$$

the points denote differentiation with respect to t.

We consider the perturbed problem $X = X_0 + X_1 + ...$ and $Y = Y_0 + Y_1 + ...$, where $X_1(t)$ and $Y_1(t)$ are the small perturbations of the Lagrangian trajectory. Then system (1) disintegrates into a chain with two first systems

$$\begin{aligned} &\dot{X}_0 = u\,(X_0, Y_0, t) \equiv u^0\,, \quad \dot{Y}_0 = v\,(X_0, Y_0, t) \equiv v^0\,; \\ &\dot{X}_1 = a_{11}X_1 + a_{12}Y_1\,, \quad \dot{Y}_1 = a_{21}X_1 + a_{22}Y_1\,, \end{aligned} \tag{2}$$

where

$$a_{11} = u_x\,(X_0, Y_0, t) \equiv u_x^0\,; \quad a_{12} = u_y^0\,; \quad a_{21} = v_x^0\,; \quad a_{22} = v_y^0 \equiv -\,a_{11}\,.$$

The elimination of Y_1 from (2) results in the equation

$$\ddot{X}_1 - \delta\dot{X}_1 - \sigma X_1 = 0\,; \tag{3}$$

here

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$$\delta = \frac{d}{dt}(\ln a_{12}) \; ; \quad \sigma = \ddot{a}_{11} + a_{11}(a_{11} - \delta) + a_{12}a_{21} \; .$$

Depending on the coefficients δ and σ, the equation (3) obtained can describe a certain oscillator. We note that the presence of arbitrary (determined or random) external perturbations in system (1) is not fundamental and it does not change the structure of (3). It is obvious, however, that the character of the dynamic system (2) is determined by the coefficients a_{ij}, which are the characteristics of the basic hydrodynamic field. Thus, analyzing the hydrodynamic field, one can evaluate the extent to which (3) characterizes oscillatory (stable or unstable) modes.

In the case where the basic field describes rotations or oscillations of the flow such an extent turns out to be large. An example is provided by the asymptotically exact solution for the dynamics of two initially point vortices in an incompressible viscous fluid [2] when the equation of perturbed motion for $X_1(t)$ turns out to be a Mathieu equation (oscillations of a pendulum with a harmonically changing suspension length) ($\sigma = a + 2q \cos 2\tau$) having unstable oscillatory modes:

$$\ddot{X}_1 + (a + 2q \cos 2\tau)\, X_1 = 0 \; , \tag{4}$$

where $\tau = 4\Gamma t / \pi \nu l^2$, and l, a, and q are defined functions slowly changing on the period cos 2τ; the points above X_1 in (4) correspond to differentiation with respect to τ.

The structure of (3)–(4) coincides with that of (1) from [1], which, in all probability, should not be interpreted as an accident since the physical scope of this analogy is quite clear — the important role of pairwise coalescence of large vortices in shear flow, which is assumed to be one governing factor of turbulent mixing in shear flows [3].

The situation with the coefficient δ is not so clear. The fact is that in examples with pronounced vortex structures, where $a_{12} \sim t^{-1} \exp(-ct^{-1})$, the coefficient δ cannot have an oscillatory character

$$\delta \sim \frac{d}{dt}(\ln a_{12}) \sim -t^{-1} + ct^{-2}$$

and its role disappears for sufficiently long times.

Thus, the reasoning presented above and being only of a qualitative illustrative character nonetheless demonstrates certain formal and physical foundations making consideration of such heuristic models quite a rational approach.

NOTATION

x, y, Eulerian coordinates; $X(t)$, $Y(t)$, Lagrangian coordinates of a liquid particle; t, time; u and v, Eulerian velocities; $a_{ij}(t)$, gradients of the basic velocity field, ij = 1 and 2; Γ, intensity (circulation) of the vortex; ν, coefficient of kinematic viscosity of the liquid; l, distance between the vortices; φ, angular deviation of the pendulum relative to the equilibrium position; a and ω, amplitude and frequency of harmonic vertical oscillations of the axis of the pendulum's suspension.

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